

BFT quantization of chiral-boson theories

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Abstract

We use the method due to Batalin, Fradkin and Tyutin (BFT) for the quantization of chiral boson theories. We consider the Floreanini-Jackiw (FJ) formulation as well as others with linear constraints.

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I. Introduction

Chiral-bosons are relevant for the understanding of several models with intrinsic chirality. Among them we find superstrings, W-gravities and general two-dimensional field theories in the light-cone. Although apparently simple, chiral-bosons present intriguing and interesting features. The great amount of work done on this subject reveals its polemic character.

One of the important points concerning chiral-bosons is related to its chirality constraint. In the FJ model [1], there is just one continuous second-class constraint [2], which contains however a zero mode that causes a time dependent symmetry in its action [3]. One alternative way of introducing the chiral constraint in the two-dimensional scalar field theory is by means of a Lagrange multiplier [4]. However, this model does not appear to be equivalent to the FJ one [5], even though both of them exhibit the same chiral constraint as classical equation of motion. In a more recent work, it has been shown how the chiral constraint can be correctly implemented by means of Lagrange multipliers in order to be equivalent to the FJ model [6,7].

All these facts make the constraint structure of chiral-boson theories an interesting subject. In addition we could also mention the Siegel formulation [8], where the chiral constraint appears as first-class. On the other hand, the quantization methods of constrained theories has been improved in these last two decades [9]. One of these methods, due to Batalin, Fradkin and Tyutin (BFT) [10,11] has as main purpose the transformation of second-class constraints into first-class ones. This is achieved with the aid of auxiliary fields that extend the phase space in a convenient way. After that, we have a gauge invariant system which matches the original theory when the so-called unitary gauge is chosen.

The BFT method is quite elegant and operates systematically. Our purpose in this paper is to introduce it in chiral-boson theories. In order to emphasize and clarify some particularities of the method, we make a brief report of it in the Sec. II. In Sec. III we apply it to the FJ chiral-boson. The main problem to be circumvented is that the BFT method

as originally introduced assumes that the system contains an even number of second-class constraints. The FJ theory has just one (continuous) constraint. So, one can take it as an infinite mode expansion that naturally gives an even (although infinite) number of second-class constraints. This way of implement the BFT quantization method over the FJ chiral-boson has been recently treated [12]. Here we consider the BFT method by directly using the continuous constraint.

In Sec. IV, we consider the chiral-boson formulated by means of linear constraint [4]. The problem mentioned above does not occur here because there is a pair of second-class constraints. The analysis of this model has also been recently reported in literature [13] where it was found two Wess-Zumino Lagrangians. We have not found the same result here. The main difference of our approaches is that we have considered the so-obtained first-class constraints in a strongly involutive way, as establishes the BFT method, and not weakly as done in ref. [13]. To conclude, we consider in Sec. V an alternative way of introducing the linear constraint in the chiral boson theory [6,7] and show that the final quantum result is equivalent to the FJ one. We devote Sec. VI to some concluding remarks.

II. Brief review of the BFT formalism

Let us consider a system described by a Hamiltonian H_0 in a phase-space (q^i, p_i) with $i = 1, \dots, N$. Let us suppose that the coordinates are bosonic (extension to include fermionic degrees of freedom and to the continuous case can be done in a straightforward way). Let us also suppose that there just exist second-class constraints (at the end of this section we refer to the case where first-class constraints are also present). Denoting them by T_a , with $a = 1, \dots, M < 2N$, we have

$$\{T_a, T_b\} = \Delta_{ab}, \quad (2.1)$$

where $\det(\Delta_{ab}) \neq 0$.

The general procedure of the BFT formalism is to convert second-class constraints into first-class ones. This is achieved by introducing auxiliary canonical variables, one for each

second-class constraint (the connection between the number of second-class constraints and the new variables in one-to-one is to keep the same number of the physical degrees of freedom in the resulting extended theory). Denoting these auxiliary variables by ψ^a we assume that they have the following general structure

$$\{\psi^a, \psi^b\} = \omega^{ab}, \quad (2.2)$$

where ω^{ab} is a constant quantity with $\det(\omega^{ab}) \neq 0$. The obtainment of these quantities is discussed in what follows. It is embodied in the obtainment of the resulting first-class constraints that we denote by \tilde{T}_a . Of course, these depend on the new variables ψ^a , namely

$$\tilde{T}_a = \tilde{T}_a(q, p; \psi) \quad (2.3)$$

and satisfy the boundary condition

$$\tilde{T}_a(q, p; 0) = \tilde{T}_a(q, p). \quad (2.4)$$

Another characteristic of these new constraints is that they are assumed to be strongly involutive, i.e.

$$\{\tilde{T}_a, \tilde{T}_b\} = 0. \quad (2.5)$$

The solution of (2.5) can be achieved by considering \tilde{T}_a expanded as

$$\tilde{T}_a = \sum_{n=0}^{\infty} T_a^{(n)}, \quad (2.6)$$

where $T_a^{(n)}$ is a term of order n in ψ . Compatibility with the boundary condition (2.4) requires that

$$T_a^{(0)} = T_a. \quad (2.7)$$

The replacement of (2.6) into (2.5) leads to a set of equations, one for each coefficient of ψ^n . We list below some of them

$$\{T_a^{(0)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(1)}\}_{(\psi)} = 0, \quad (2.8a)$$

$$\{T_a^{(0)}, T_b^{(1)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(2)}\}_{(\psi)} + \{T_a^{(2)}, T_b^{(1)}\}_{(\psi)} = 0, \quad (2.8b)$$

$$\begin{aligned} \{T_a^{(0)}, T_b^{(2)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(1)}\}_{(q,p)} + \{T_a^{(2)}, T_b^{(0)}\}_{(q,p)} + \{T_a^{(1)}, T_b^{(3)}\}_{(\psi)} \\ + \{T_a^{(2)}, T_b^{(2)}\}_{(\psi)} + \{T_a^{(3)}, T_b^{(1)}\}_{(\psi)} = 0, \end{aligned} \quad (2.8c)$$

\vdots

These correspond to coefficients of the powers $\psi^0, \psi^1, \psi^2, \dots$, respectively. The notation used above, $\{, \}_{(q,p)}$ and $\{, \}_{(\psi)}$, represent the parts of the Poisson bracket $\{, \}$ relative to the variables (q, p) and (ψ) .

Equations (2.8) are used iteratively in the obtainment of the corrections $T^{(n)}$ ($n \geq 1$). The first equation (2.8) shall give $T^{(1)}$. With this result and (2.8b), one calculates $T^{(2)}$, and so on. Since $T^{(1)}$ is linear in ψ we may write

$$T_a^{(1)} = X_{ab}(q, p) \psi^b. \quad (2.9)$$

Introducing this expression into (2.8a) and using the boundary condition (2.4), as well as (2.1) and (2.2), we get

$$\Delta_{ab} + X_{ac} \omega^{cd} X_{bd} = 0. \quad (2.10)$$

We notice that this equation does not give X_{ab} univocally, because it also contains the still unknown ω^{ab} . What we usually do is to choose ω^{ab} in such a way that the new variables are

unconstrained (*). The knowledge of X_{ab} permits us to obtain $T_a^{(1)}$. If X_{ab} do not depend on (q, p) , it is easily seen that $T_a + T_a^{(1)}$ is already strongly involutive. When this occurs, we are succeed in obtaining \tilde{T}_a . If this is not so we have to introduce $T_a^{(1)}$ into (2.8b) to calculate $T_a^{(2)}$, and so on.

Another point in the BFT formalism is that any dynamical function $A(q, p)$ (for instance the Hamiltonian) has also to be properly modified in order to be strongly involutive with the first-class constraints \tilde{T}_a . Denoting the modified quantity by $\tilde{A}(q, p; \psi)$, we then have

$$\{\tilde{T}_a, \tilde{A}\} = 0. \quad (2.11)$$

In addition, \tilde{A} has also to satisfy the boundary condition

$$\tilde{A}(q, p; 0) = A(q, p). \quad (2.12)$$

The obtainment of \tilde{A} is similar to what was done to get \tilde{T}_a , that is to say, we consider an expansion like

$$\tilde{A} = \sum_{n=0}^{\infty} A^{(n)}, \quad (2.13)$$

where $A^{(n)}$ is also a term of order n in ψ' s. Consequently, compatibility with (2.12) requires that

$$A^{(0)} = A. \quad (2.14)$$

(*) It is opportune to mention that this procedure is not always possible to be done. We shall return to this point in the examples to be discussed in the next sections.

The combination of (2.6), (2.11) and (2.13) gives

$$\{T_a^{(0)}, A^{(0)}\}_{(q,p)} + \{T_a^{(1)}, A^{(1)}\}_{(\psi)} = 0, \quad (2.15a)$$

$$\{T_a^{(0)}, A^{(1)}\}_{(q,p)} + \{T_a^{(1)}, A^{(0)}\}_{(q,p)} + \{T_a^{(1)}, A^{(2)}\}_{(\psi)} + \{T_a^{(2)}, A^{(1)}\}_{(\psi)} = 0, \quad (2.15b)$$

$$\begin{aligned} \{T_a^{(0)}, A^{(2)}\}_{(q,p)} + \{T_a^{(1)}, A^{(1)}\}_{(q,p)} + \{T_a^{(2)}, A^{(0)}\}_{(q,p)} + \{T_a^{(1)}, A^{(3)}\}_{(\psi)} \\ + \{T_a^{(2)}, A^{(2)}\}_{(\psi)} + \{T_a^{(3)}, A^{(1)}\}_{(\psi)} = 0, \end{aligned} \quad (2.15c)$$

\vdots

which correspond to the coefficients of the powers ψ^0, ψ^1, ψ^2 , etc., respectively. The first expression above gives us $A^{(1)}$

$$A^{(1)} = -\psi^a \omega_{ab} X^{bc} \{T_c, A\}, \quad (2.16)$$

where ω_{ab} and X^{ab} are the inverse of ω^{ab} and X_{ab} .

In the obtainment of $T_a^{(1)}$ we had seen that $T_a + T_a^{(1)}$ was strongly involutive if the coefficients X_{ab} do not depend on (q, p) . However, the same argument does not necessarily apply here. It might be necessary to calculate other corrections to obtain the final \tilde{A} . Let us discuss how this can be systematically done. We consider the general case first. The correction $A^{(2)}$ comes from (2.15b), that we conveniently rewrite as

$$\{T_a^{(1)}, A^{(2)}\}_{(\psi)} = -G_a^{(1)}, \quad (2.17)$$

where

$$G_a^{(1)} = \{T_a, A^{(1)}\}_{(q,p)} + \{T_a^{(1)}, A\}_{(q,p)} + \{T_a^{(2)}, A^{(1)}\}_{(\psi)}. \quad (2.18)$$

Thus

$$A^{(2)} = -\frac{1}{2} \psi^a \omega_{ab} X^{bc} G_c^{(1)}. \quad (2.19)$$

In the same way, other terms can be obtained. The final general expression reads

$$A^{(n+1)} = -\frac{1}{n+1} \psi^a \omega_{ab} X^{bc} G_c^{(n)}, \quad (2.20)$$

where

$$G_a^{(n)} = \sum_{m=0}^n \{T_a^{(n-m)}, A^{(m)}\}_{(q,p)} + \sum_{m=0}^{n-2} \{T_a^{(n-m)}, A^{(m+2)}\}_{(\psi)} + \{T_a^{(n+1)}, A^{(1)}\}_{(\psi)}. \quad (2.21)$$

For the particular case when X_{ab} do not depend on (q, p) we have that the corrections $A^{(n+1)}$ can be obtained by the same expression (2.20), but $G_a^{(n)}$ simplifies to

$$G_a^{(n)} = \{T_a, A^{(n)}\}_{(q,p)} + \{T_a^{(1)}, A^{(n-1)}\}_{(q,p)}. \quad (2.22)$$

To conclude this brief report on the BFT formalism, we refer to the case where there are also first-class constraints. Let us call them by F_α . We consider that the constraints of the theory satisfy the following involutive algebra (with the use of the Dirac bracket definition to strongly eliminate the second-class constraints)

$$\{F_\alpha, F_\beta\}_D = U_{\alpha\beta}^\gamma F_\gamma + I_{\alpha\beta}^a T_a, \quad (2.23a)$$

$$\{H_0, F_\alpha\}_D = V_\alpha^\beta F_\beta + K_\alpha^a T_a, \quad (2.23b)$$

$$\{F_\alpha, T_b\}_D = 0. \quad (2.23c)$$

In (2.23), $U_{\alpha\beta}^\gamma$, $I_{\alpha\beta}^a$, V_α^β and K_α^a are structure functions of the involutive algebra.

The BFT procedure in this case also introduces one auxiliary variable for each one of the second-class constraints (this is also in agreement with the counting of the physical

degrees of freedom of the initial theory). All the constraints and the Hamiltonian have to be properly modified in order to satisfy the same involutive algebra above, namely,

$$\{\tilde{F}_\alpha, \tilde{F}_\beta\}_D = U_{\alpha\beta}^\gamma \tilde{F}_\gamma + I_{\alpha\beta}^a \tilde{T}_a, \quad (2.24a)$$

$$\{\tilde{H}_0, \tilde{F}_\alpha\}_D = V_\alpha^\beta \tilde{F}_\beta + K_\alpha^a \tilde{T}_a, \quad (2.24b)$$

$$\{\tilde{F}_\alpha, \tilde{T}_b\}_D = 0. \quad (2.24c)$$

Since the algebra is now weakly involutive, the iterative calculation of the previous case cannot be applied here. We have to figure out the corrections that have to be done in the initial quantities.

III. The Floreanini-Jackiw chiral boson

This is described by the following Lagrangian density [1]

$$\mathcal{L} = \dot{\phi} \phi' - \phi'^2, \quad (3.1)$$

where dot and prime represent derivatives with respect time and space coordinates respectively. Space-time is assumed to be a two-dimensional Minkowskian variety. The chiral condition $\dot{\phi} - \phi' = 0$ is obtained as an equation of motion up to a general function of time. The canonical momentum conjugate to ϕ is

$$\pi = \phi'. \quad (3.1)$$

This is a constraint that we denote by

$$T(\phi, \pi) = \pi - \phi'. \quad (3.2)$$

We construct the primary Hamiltonian density

$$\begin{aligned}
\mathcal{H} &= \pi \dot{\phi} - \mathcal{L} + \xi T, \\
&= (\pi - \phi') \dot{\phi} + \phi'^2 + \xi (\pi - \phi'), \\
&\rightarrow \phi'^2 + \xi (\pi - \phi'),
\end{aligned} \tag{3.3}$$

where in the last step we have absorbed the velocity $\dot{\phi}$ in the Lagrange multiplier ξ . The consistency condition for the constraint T does not lead to any new one.

The constraint above is second-class, in a sense that they satisfy the Poisson bracket relation

$$\{T(x), T(y)\} = -2 \delta'(x - y). \tag{3.4}$$

This bracket and those that follow are taken at the same time, $x_0 = y_0$.

To implement the BFT formalism we have to introduce an auxiliary field ψ satisfying the following general bracket structure

$$\{\psi(x), \psi(y)\} = \omega(x, y), \tag{3.5}$$

where ω is antisymmetric in x, y . Of course, this cannot be achieved in terms of Poisson brackets. There are two ways (maybe more) to circumvent this problem. One of them is considering all fields expanded in terms of Fourier modes. This leads to a infinite number of variables and it is possible to have an expression like (3.5) but in terms of Poisson brackets. As it was told in the introduction, we have discussed this case in a previous paper [12]. The second procedure is to consider that ψ is constrained and that expression (3.5) is realized in terms of Dirac brackets. We shall follow this second possibility here.

We now extend the constraint T to \tilde{T}

$$\tilde{T} = \tilde{T}(\phi, \pi; \psi), \tag{3.6}$$

with the boundary condition

$$\tilde{T}(\phi, \pi; 0) = T \quad (3.7)$$

and consider that they are strong involutive, i.e.

$$\{\tilde{T}(x), \tilde{T}(y)\} = 0. \quad (3.8)$$

The obtainment of \tilde{T} follows the procedure discussed in Sec. 2. We first have to solve the equation (see expression 2.10)

$$\int dz dr X(x, z) w(z, r) X(y, r) = -\Delta(x, y), \quad (3.9)$$

where $\Delta(x, y)$ is related to the structure given by (3.4), i.e.

$$\Delta(x, y) = -2\delta'(x - y). \quad (3.10)$$

Since we are considering that the bracket involving ψ 's is constrained, let us choose

$$\omega(x, y) = 2\delta'(x - y). \quad (3.11)$$

In consequence, the solution of (3.9) gives

$$X(x, y) = \delta(x - y). \quad (3.12)$$

We notice that the quantity X does not depend on the initial fields (ϕ, π) . This means that

$$\begin{aligned} \tilde{T}(x) &= T(x) + T^{(1)}(x), \\ &= T(x) + \int dy X(x, y) \psi(y), \\ &= T(x) + \psi(x). \end{aligned} \quad (3.13)$$

In fact, one can easily verify that $\tilde{T} = T + \psi$ satisfies the involutive expression (3.8).

We now pass to consider the obtainment of \tilde{H}_c . Considering what we have seen in Sec. 2 and that we also have $T^{(n)} = 0$ for $n \geq 2$, the corrections that give \tilde{H}_c can be written as

$$H_c^{(n+1)} = -\frac{1}{n+1} \int dx dy dz \, \psi(x) \omega^{-1}(x, y) X^{-1}(y, z) G^{(n)}(z), \quad (3.14)$$

where $G^{(n)}(x)$ is given by

$$G^{(n)}(x) = \{T(x), H_c^{(n)}\}_{(\phi, \pi)} + \{T^{(1)}(x), H_c^{(n-1)}\}_{(\phi, \pi)} \quad (3.15)$$

and

$$\begin{aligned} \omega^{-1}(x, y) &= \frac{1}{2} \theta(x - y), \\ X^{-1}(x, y) &= \delta(x - y). \end{aligned} \quad (3.16)$$

The quantity $\theta(x - y)$ that appears in (3.16) is the usual theta-function. The initial canonical Hamiltonian can be figure out from (3.3) as

$$H_c = \int dx \, \phi'^2. \quad (3.17)$$

Thus, using (3.15) and (3.17) we get

$$G^{(0)}(x) = 2 \phi''. \quad (3.18)$$

The combination of (3.14), (3.17) and (3.18) permit us to calculate $H_c^{(1)}$

$$H_c^{(1)} = - \int dx \, \psi \phi'. \quad (3.19)$$

The next corrections are

$$G^{(1)} = \psi'(x) , \quad (3.20)$$

$$H_c^{(2)} = \frac{1}{4} \int dx \psi^2(x) . \quad (3.21)$$

It is easily seen that other corrections are zero. The extended canonical Hamiltonian density is then given by

$$\tilde{\mathcal{H}}_c = \phi'^2 - \psi \phi' + \frac{1}{4} \psi^2 . \quad (3.22)$$

In fact, one can check that it has strong involution with the constraint \tilde{T} .

We now look for the Lagrangian that leads to this extended theory. A consistent way of doing this is by means of the path integral formalism in the Faddeev-Senjanovic procedure [14,15]. Since ψ is constrained, it is not difficult to see that the constraint that leads to the bracket (3.5) with ω given by (3.11) is

$$R = p'_\psi + \frac{1}{4} \psi , \quad (3.23)$$

where p_ψ is the canonical momentum related to ψ . The general expression of the vacuum functional then reads

$$Z = N \int [d\mu] \exp \left\{ i \int d^2x \left(p\dot{\lambda} + \pi\dot{\phi} + p_\psi\dot{\psi} - \tilde{\mathcal{H}}_c \right) \right\} , \quad (3.24)$$

with the measure $[d\mu]$ given by

$$[d\mu] = [d\phi][d\psi][d\pi][dp_\psi] \delta[\pi - \psi' + \psi] \delta[p'_\psi + \frac{1}{4}\psi] \delta[\tilde{\chi}] |\det\{\cdot\}|^{1/2} \quad (3.25)$$

and $\tilde{\chi}$ is the gauge-fixing function corresponding to the first-class constraint \tilde{T} . The term $|\det\{\cdot\}|$ is representing the determinant of all constraints of the theory, including the

gauge-fixing ones. The integration over π is easily done by using the first delta functional and the integration over p_ψ gives a nonlocal term. The final result is

$$Z = N \int [d\phi][d\psi] \delta[\chi] |\det\{\cdot, \cdot\}|^{1/2} \exp \left\{ i \int d^2x \left[\dot{\phi}\phi' - \phi'^2 + \psi (\phi' - \dot{\phi}) - \frac{1}{4} \psi^2 - \frac{1}{4} \dot{\psi} \int dy \theta(x-y) \psi(y) \right] \right\}. \quad (3.26)$$

From the expression above we obtain that the Lagrangian density that leads to the extended theory we have discussed is

$$\mathcal{L} = \dot{\phi}\phi' - \phi'^2 + \psi (\phi' - \dot{\phi}) - \frac{1}{4} \psi^2 - \frac{1}{4} \dot{\psi} \int dy \theta(x-y) \psi(y). \quad (3.27)$$

As one observes, it leads to the usual FJ Lagrangian of the chiral-boson theory when the auxiliary field ψ is turned off. Incidentally we mention that this is the same Lagrangian obtained from Fourier modes expansion as discussed in reference [12].

It may be borrowing us the presence of a second-class constraint in the implementation of the BFT formalism. This constraint could also be transformed into a first-class one by introducing a new auxiliary field, also constrained. This last constraint can also be transformed into first-class in a endless process [16].

IV. Chiral-bosons with linear constraint

The chiral-boson condition $\dot{\phi} - \phi' = 0$ can also be introduced in the two-dimensional scalar field theory by means of a Lagrange multiplier [4] (*)

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \lambda (\dot{\phi} - \phi'). \quad (4.1)$$

(*) We use the metric convention: $\eta_{00} = -\eta_{11} = -1$, $\eta_{01} = 0$.

This theory has been criticized by some authors [5]. The main arguments are that it does not lead to a positive definite Hamiltonian and that its physical spectrum is just the vacuum state. In fact, the theory described by the Lagrangian (4.1) is not equivalent to the FJ one, even though both of them contain the same chiral condition $\dot{\phi} - \phi' = 0$ as classical equation of motion. We are going to study this point with details by means of the BFT formalism. In the next section we shall discuss what is missing in the Lagrangian (4.1) to correct describe the usual chiral-boson theory.

From the Lagrangian (4.1) we get the momenta

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0, \quad (4.2)$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} + \lambda. \quad (4.3)$$

Expression (4.2) is a (primary) constraint. We then construct the primary Hamiltonian

$$\mathcal{H} = \frac{1}{2} (\pi^2 + \phi'^2 + \lambda^2) + \lambda(\phi' - \pi) + \xi p, \quad (4.4)$$

where the velocity $\dot{\lambda}$ was absorbed by ξ . The consistency condition leads to a new constraint

$$\pi - \phi' - \lambda = 0. \quad (4.5)$$

Let us denote these constraints by

$$T_1 = p, \quad (4.6)$$

$$T_2 = \pi - \phi' - \lambda. \quad (4.7)$$

They are second-class and their nonvanishing brackets read

$$\begin{aligned}\{T_1(x), T_2(y)\} &= \delta(x - y), \\ \{T_2(x), T_2(y)\} &= -2\delta'(x - y).\end{aligned}\tag{4.8}$$

From these results, we can see that, in fact, Lagrangian (4.1) cannot lead to the same theory of the FJ one. The former has two degrees of freedom given by the fields ϕ and its momentum π and one second-class constraint. So, it has just one physical degree of freedom. The case with linear constraint has four degrees of freedom, related to ϕ , π , λ and p and two second class constraints. So, differently from the first case, it has two physical degrees of freedom.

To implement the BFT formalism, we introduce two auxiliary fields ψ^a , with $a=1,2$, satisfying the following structure

$$\{\psi^a(x), \psi^b(y)\} = \omega^{ab}(x, y),\tag{4.9}$$

where $\omega^{ab}(x, y)$ is also antisymmetric in a, b . We thus extend the constraints T_a to \tilde{T}_a such that

$$\tilde{T}_a = \tilde{T}_a(\phi, \pi, \lambda, p; \psi^a),\tag{4.10}$$

with the usual boundary condition

$$\tilde{T}_a(\phi, \pi, \lambda, p; 0) = T_a\tag{4.11}$$

and consider that they are strong involutive

$$\{\tilde{T}_a, \tilde{T}_b\} = 0.\tag{4.12}$$

The obtainment of \tilde{T}_a is given as discussed in Sec. 2 and in the previous section. First, we have to solve the equation

$$\int dzdr X_{ac}(x, z) \omega^{cd}(z, r) X_{bd}(y, r) = -\Delta_{ab}(x, y), \quad (4.13)$$

where Δ_{ab} is related to the structure of the Poisson brackets of the constraints T_a . From (4.8), we get (*)

$$\Delta_{ab} = (\epsilon_{ab} - 2\delta_{a2} \delta_{b2} \partial_x) \delta(x - y). \quad (4.14)$$

Here, it is possible to consider that the auxiliary variables ψ^a are not constrained. We thus take

$$\omega^{ab}(x, y) = \epsilon^{ab} \delta(x - y), \quad (4.15)$$

that is to say, we are considering that one of the ψ^a is the canonical momentum of the other. The bracket (4.9), where ω^{ab} is given by (4.15), is in a unconstrained symplectic form. Introducing (4.14) and (4.15) into (4.13) we get

$$X_{ab}(x, y) = (\epsilon_{ab} - \delta_{a2} \delta_{b2} \partial_x) \delta(x - y). \quad (4.16)$$

We notice that X_{ab} do not depend on the initial fields ϕ , λ , π and p . This means that the extended first-class constraints \tilde{T}_a are just given by

$$\begin{aligned} \tilde{T}_a(x) &= T_a(x) + T_a^{(1)}(x), \\ &= T_a(x) + \int dy X_{ab}(x, y) \psi^b(y). \end{aligned} \quad (4.17)$$

(*) $\epsilon_{12} = -\epsilon^{12} = 1$.

Hence,

$$\tilde{T}_1 = p + \psi^2, \quad (4.18)$$

$$\tilde{T}_2 = \pi - \phi' - \lambda - \psi^1 - \psi^{2'}. \quad (4.19)$$

One can actually see that \tilde{T}_1 and \tilde{T}_2 satisfy the strong involution relation (4.12).

Considering what was done in the previous section, the corrections that give \tilde{H}_C are

$$H_C^{(n+1)} = -\frac{1}{n+1} \int dx dy dz \psi^a(x) \omega_{ab}(x, y) X^{bc}(y, z) G_c^{(n)}(z), \quad (4.20)$$

where $G_a^{(n)}(x)$ are

$$G_a^{(n)}(x) = \{T_a(x), H_C^{(n)}\}_{(\phi, \lambda, \pi, p)} + \{T_a^{(1)}(x), H_C^{(n-1)}\}_{(\phi, \lambda, \pi, p)} \quad (4.21)$$

and the inverses ω_{ab} and X^{ab} read

$$\omega_{ab}(x, y) = \epsilon_{ab} \delta(x - y), \quad (4.22)$$

$$X^{ab}(x, y) = (\epsilon^{ab} - \delta^{a1} \delta^{b1} \partial_x) \delta(x - y). \quad (4.23)$$

As the canonical Hamiltonian is (see expression 4.4)

$$\mathcal{H}_C = \frac{1}{2} (\pi^2 + \phi'^2 + \lambda^2) + \lambda (\phi' - \pi), \quad (4.24)$$

we obtain

$$G_1^{(0)} = T_2, \quad (4.25)$$

$$G_2^{(0)} = -T_2' + \lambda', \quad (4.26)$$

$$\mathcal{H}_C^{(1)} = \psi^1 T_2 + \psi^2 \lambda', \quad (4.27)$$

$$G_1^{(1)} = -\psi^1 - \psi^{2'}, \quad (4.28)$$

$$G_2^{(1)} = 2\psi^{1'}(x), \quad (4.29)$$

$$\mathcal{H}_C^{(2)} = \frac{1}{2}(\psi^1 \psi^1 + 2\psi^1 \psi^{2'} + \psi^2 \psi^{2''}). \quad (4.30)$$

Other corrections are zero. We can thus write the extended canonical Hamiltonian

$$\begin{aligned} \tilde{H}_C &= H_C + H_C^{(1)} + H_C^{(2)}, \\ &= \int dx \left[\frac{1}{2} (\pi^2 + \phi'^2 + \lambda^2) + \lambda(\phi' - \pi) - \psi^1(\pi - \phi' - \lambda) - \psi^2 \lambda' \right. \\ &\quad \left. + \frac{1}{2} (\psi^1 \psi^1 + 2\psi^1 \psi^{2'} + \psi^2 \psi^{2''}) \right]. \end{aligned} \quad (4.31)$$

It is also a direct check to verify that \tilde{H}_C is, in fact, also strongly involutive with the constraints \tilde{T}_a .

In a previous analysis of the chiral-boson theory with linear constraint, Miao, Zhou and Liu [13] have found two WZ Hamiltonians, contrarily to what we have found here. The difference between our results is mainly because they have started with weakly involutive relations instead of strong ones.

Finally, we look for the Lagrangian that leads to this extended theory. We are now considering that ψ^a are not constrained. From the symplectic form $\{\psi^a, \psi^b\} = \epsilon^{ab}$ we can write them as the canonical pair

$$\begin{aligned} \psi^1 &= p_\psi, \\ \psi^2 &= \psi. \end{aligned} \quad (4.32)$$

Here, as there are just first-class constraints, the general expression of the vacuum functional follows from the Faddeev path integral formulation [14], i.e.

$$Z = N \int [d\mu] \exp \left\{ i \int d^2x \left(p\dot{\lambda} + \pi\dot{\phi} + p_\psi\dot{\psi} - \tilde{\mathcal{H}} \right) \right\}, \quad (4.33)$$

where the measure $[d\mu]$ reads

$$[d\mu] = [d\phi][d\pi][d\lambda][dp][d\psi][dp_\psi] |\det\{\tilde{T}_a, \tilde{\chi}_a\}| \prod_{a=1}^2 \delta[\tilde{T}_a] \delta[\chi_a]. \quad (4.34)$$

The quantities $\tilde{\chi}_a$ are gauge-fixing constraints (in the Faddeev procedure, they have to satisfy $\{\tilde{\chi}_a, \tilde{\chi}_b\} = 0$).

The effective Lagrangian is then obtained by integrating over the momenta. The use of the delta functionals makes the integration over p and π quite trivial. Integration over p_ψ gives another delta functional (in this last step we assume that the gauge-fixing constraints do not contain p_ψ). The result is

$$Z = N \int [d\phi][d\lambda][d\psi] \delta[\dot{\phi} - \phi' + \dot{\psi} - \psi'] \exp \left\{ i \int d^2x \left[\psi(\lambda' - \dot{\lambda}) + (\psi' - \dot{\psi})(\phi' + \lambda + \psi') \right] \right\}, \quad (4.35)$$

where it was used the expression of the delta functional in the effective action.

One observes once more that the chiral boson theory with linear constraint as given by (4.1) is not effectively equivalent to the FJ one. For instance, if one turns out the auxiliary field one obtains an identically null Lagrangian. This kind of problem had already been pointed out in a previous paper [6].

An interesting point is that even though the effective Lagrangian in (4.35) leads to an identically null theory when the auxiliary field is removed, we can verify that it also leads to the model we have here discussed, that is to say, with the first-class constraints \tilde{T}_a .

V. Chiral-bosons with linear constraint revisited

There is another way of introducing the linear constraint in the chiral-boson theory. Instead of the Lagrangian (4.1) we take [6,7]

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \lambda (\dot{\phi} - \phi') + \frac{1}{2} \lambda^2. \quad (5.1)$$

The canonical momenta are the same as in the other case, namely

$$p = 0, \quad (5.2)$$

$$\pi = \dot{\phi} + \lambda. \quad (5.3)$$

The relation (5.2) is a (primary) constraint. Consistency conditions lead to secondary and tertiary constraints. These are

$$\pi - \phi' = 0 \quad (5.4),$$

$$\lambda' = 0. \quad (5.5)$$

It is important to mention that constraint (5.5) was just obtained by virtue of the extra term $\lambda^2/2$ of the Lagrangian (5.1). We rewrite all the constraints above as

$$T_1 = p, \quad (5.6a)$$

$$T_2 = \pi - \phi', \quad (5.6b)$$

$$T_3 = \lambda'. \quad (5.6c)$$

These are second-class and satisfy the algebra

$$\begin{aligned}\{T_1(x), T_3(y)\} &= \delta'(x - y), \\ \{T_2(x), T_2(y)\} &= -2\delta'(x - y).\end{aligned}\tag{5.7}$$

It is opportune to mention that this theory, contrarily to what occurs with the one discussed in Sec. IV, has just one physical degree of freedom (the same of the FJ theory).

The direct use of the BFT procedure, as it was discussed in the previous sections, leads to new (first-class) constraints and a new canonical Hamiltonian. We just write them below

$$\tilde{T}_1 = p + p_\chi, \tag{5.8a}$$

$$\tilde{T}_2 = \pi - \phi' + \psi, \tag{5.8b}$$

$$\tilde{T}_3 = \lambda' - \chi', \tag{5.8c}$$

$$\tilde{\mathcal{H}}_C = \frac{1}{2}(\pi^2 + \phi'^2) + (\chi - \lambda)(\pi - \phi' + \psi) + \frac{1}{2}\psi(\pi - \phi') + \frac{1}{4}\psi^2. \tag{5.9}$$

The auxiliary fields we have introduced are χ and its momentum p_χ plus a constrained field ψ , which we have considered to satisfy the same bracket structure given by (3.5) and (3.11). Using the path integral formalism in the Faddeev-Senjanovic procedure, we get

$$\begin{aligned}Z = N \int [d\phi][d\psi]\delta[\tilde{\chi}] \det |\{, \}| \exp \Big\{ i \int d^2x \Big[\phi' \dot{\phi} - \phi'^2 + \psi(\phi' - \dot{\phi}) - \frac{1}{4}\psi^2 \\ + \frac{1}{4}\dot{\psi} \int dy \theta(x - y)\psi(y) \Big] \Big\}. \end{aligned} \tag{5.10}$$

We see that the effective Lagrangian extracted from (5.10) is exactly the same as (3.27).

VI. Conclusion

We have studied chiral-boson theories by means of the BFT quantization method. First we have considered the FJ formulation, where we had to use just one constrained auxiliary variable in order to convert the (continuous) second-class constraint into first-class. As a result we have obtained a nonlocal Lagrangian, which is in agreement with a previous treatment by means of Fourier modes expansion. Secondly we have dealt with the case with linear constraint. We have obtained an effective Lagrangian that does not lead to the FJ one when the auxiliary fields are turned off. This fact may be reflecting the inconsistencies of the model as it was initially formulated. Finally, we have considered an improved way of introducing the linear constraint in the chiral-boson theory. We have shown that the inconsistencies have disappeared in this last situation and that the BFT treatment leads to same result of the FJ case.

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